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Extraction of Normal Modes and Full Modal Damping from Complex Modal Parameters

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A procedure for extracting the structural normal modes and nondiagonal damping matrix from damped system realization parameters is presented. The procedure utilizes the state-space realization of experimental data obtained using identification algorithms such as the Eigensystem Realization Algorithm and Polyreference. This realization is first decomposed into the complex damped modes of the system and each mode is then transformed into an equivalent real-valued approximation of a second-order system. The transformation yields an approximation of the normal mode frequencies, mode shapes, and damping ratios. The procedure then develops a corrective transformation that accounts for the modal coupling implicit in nonproportional damping, thus yielding the correct structural normal modes and a nondiagonal modal damping matrix for the realization. Solutions for the corrective transformation are shown to be subject to a displacement consistency criterion. The damping identification procedure is demonstrated for both overdetermined (more sensors than modes) and underdetermined (more modes than sensors) problems using simulated data.

I. Introduction

THE problem of extracting the normal modes from the parameters of a nonproportionally damped structural dynamic system realization remains a challenge to the structural analyst. A nonproportionally damped system is one for which the undamped and damped mode shapes are not equivalent, and thus the undamped modes, which are characteristic of the purely oscillatory behavior of the system, are not directly identifiable from the test-measured modal parameters. Because the primary aim of modal testing is to construct or verify mass and stiffness models without knowledge of the damping, the modal parameters used to validate mass and stiffness remain exactly valid only for proportionally damped systems. Most physical systems are not proportionally damped, however, except in the limit of very light damping. In addition, closed-loop controlled structures are typically characterized by significant damping. As such, closed-loop identification of the structural system must account for nonproportional damping if the resulting mass and stiffness are to yield the desired normal modes.

The objective of the present paper is to fully develop a transformation-theory-based structural identification algorithm applicable to multi-input/multi-output (MIMO) realizations of structural systems that exhibit nonproportional damping. The method presented herein builds on recent developments in transformation theory for structural identification. A discrete-time state-space realization of modal test data is obtained using the Eigensystem Realization Algorithm (ERA)[1,2] or Polyreference,[3] resulting in a set of difference equations. The realization is converted to continuous time, yielding a nominal first-order model. To obtain the normal modes, the model first is transformed to a canonical basis via the Common Basis Structural Identification (CBSI) algorithm.[4] It has been shown that the resulting model renders a unique set of frequencies and normal mode shapes for proportionally damped cases. However, for nonproportionally damped cases, the CBSI-transformed basis, by assuming a diagonal modal damping matrix, becomes at best a pseudonormal basis. In other words, it does not yield the true normal mode frequencies, mode shapes, and full modal damping matrix.

A transformation then is proposed for correcting the pseudonormal basis to determine the corrected normal modal parameters. Constraints on this transformation are derived that ensure the physical consistency of the resulting model. The defined transformation and constraints allow for a direct solution in the well-determined and overdetermined cases (where the number of sensors equal or exceed the number of modes) that requires only as many equations as the number of modes. An extension of the direct solution is also possible when the number of sensors is less than the number of modes. A scaling matrix is determined iteratively that augments a pseudoinverse solution to satisfy the transformation constraints.

In contrast, previous work in this area has focused only on problems in which the number of sensor measurements is equal to or greater than the number the identified modes.[5–10] Ibrahim's[5,6] methods explicitly transform the damped modes to a physical basis consisting of the measured coordinates, whereas Zhang and Lallement[7,8] iteratively determined a complex transformation to express the damped modes in a basis of estimated normal-modes variables. Placidi et al.[9] reviewed these existing methods and their deficiencies and proposed an improved iterative algorithm that is independent of the mode-shape normalization and allows for more sensors than measured modes. Minas and Inman[10] determined a direct-solution approach, but the system of equations is large in comparison to the dimension of the system, and the determinacy of the solution is not clearly apparent. Yang and Yeh[11] presented a method for direct determination of mass, damping, and stiffness, but their approach was limited to well-determined problems with equal numbers of modes, sensors, and actuators, an impractical restriction. Finally, Hasselman[12] studied the use of a known mass matrix with complex modes to decouple the mass, stiffness, and damping effects. This approach, however, requires accurate knowledge of the system mass matrix.

For modern structural system identification, problems arise involving flexible space structures that possess a large number of modes within the bandwidth of modal test instrumentation. Advances in modern computational methods and hardware have enabled the analyst to accurately fit the measured data of MIMO modal tests. These algorithms now can compute models possessing more identified modes than the number of sensor measurements. Hence, there remains a need to develop a general structural identification and damping correction method that handles both

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underdetermined problems (where fewer sensors than modes are available) and determines the resultant system in as direct and computationally efficient manner as possible. Furthermore, noise and numerical effects that corrupt the experimental damped modal parameters can significantly impact the quality of existing techniques that do not consider the physical consistency of the resultant model. The procedure presented herein is designed to provide a more general and robust approach to the damping correction problem.

To this end, the remainder of the paper is organized as follows: Section II reviews the inverse damped vibration problem, which is a theoretical basis for the damping correction problem. Section III reviews system realization models and the CBSI procedure for extracting estimated normal modes. The basis correction transformation for nonproportional damping then is presented in Sec. IV. Section V details the implementation and performance of the damping correction algorithm on a series of numerical examples. Concluding remarks are offered in Sec. VI, and some algorithmic details are presented in the Appendix.

II. Inverse Damped Vibration Problem

We begin by reviewing the damped equations of motion for linear structural dynamics, the governing symmetric generalized eigenproblem, and the resulting inverse problem whereby measured vibration parameters are transformed to physical system quantities such as mass, damping, and stiffness matrices. As noted previously, all methods for relating the nonproportionally damped and nominally undamped vibration modes are founded on the solution to the inverse damped vibration problem.

In terms of a chosen set of n physical displacements $q(t)$, the equations of motion are given as

$$M\ddot{q} + D\dot{q} + Kq = \hat{B}u \tag{1}$$

where M, D, and K are the $n \times n$ inertia, damping, and stiffness matrices, respectively, and u is an m-input force vector. When D is given as a general symmetric positive semidefinite matrix, a symmetrical firstorder form of Eq. (1) can be written using a canonical form, viz.,

$$\begin{bmatrix} D & M \\ M & 0 \end{bmatrix} \begin{Bmatrix} \dot{q} \\ \ddot{q} \end{Bmatrix} = \begin{bmatrix} -K & 0 \\ 0 & M \end{bmatrix} \begin{Bmatrix} q \\ \dot{q} \end{Bmatrix} + \begin{bmatrix} \hat{B} \\ 0 \end{bmatrix} u \tag{2}$$

Thus, the generalized eigenproblem can be written in a symmetric form as

$$\begin{bmatrix} -K & 0 \\ 0 & M \end{bmatrix} \begin{bmatrix} X \\ X\Lambda \end{bmatrix} = \begin{bmatrix} D & M \\ M & 0 \end{bmatrix} \begin{bmatrix} X \\ X\Lambda \end{bmatrix} \Lambda \tag{3}$$

such that

$$\begin{bmatrix} X \\ X\Lambda \end{bmatrix}^T \begin{bmatrix} D & M \\ M & 0 \end{bmatrix} \begin{bmatrix} X \\ X\Lambda \end{bmatrix} = I$$

$$\begin{bmatrix} X \\ X\Lambda \end{bmatrix}^T \begin{bmatrix} -K & 0 \\ 0 & M \end{bmatrix} \begin{bmatrix} X \\ X\Lambda \end{bmatrix} = \Lambda \tag{4}$$

$$\Lambda = \mathrm{diag}\{\sigma_i \pm j\omega_i, \quad i = 1, \ldots, n\}$$

$$X = [\ldots \Re(X_i) \pm j\Im(X_i) \ldots]$$

The columns of X are the complex damped mode shapes that may possess complicated phase relationships within each decoupled mode between spatially distinct points. If a proportionally damped system is expressed in this form, the relative phase relationships are either 0 or π radians, implying that the real and imaginary parts of X_i describe the same deformation; in other words, the displacement and velocity mode shapes are equivalent. Otherwise, nonnormal phase values indicate the presence of nonproportional damping, such that the mode shapes of the undamped system and the damped system are distinctly different quantities.

The symmetric formulation equations (3) and (4) lead to a general solution for the inverse damped vibration problem when Λ and X, measured at all n points in q and normalized as in Eq. (4), are known:

$$K^{-1} = -X\Lambda^{-1}X^T \qquad M^{-1} = X\Lambda X^T \qquad D = -M(X\Lambda^2 X^T)M \tag{5}$$

together with the condition that

$$XX^T = 0 \tag{6}$$

The solution of the preceding inverse damped vibration problem yields two important results of interest to the nonproportional damping correction problem. First, it effectively decouples the influence of the general damping matrix D from the stiffness and mass, which in turn possess the properties of the normal undamped vibrational modes. Second, Eq. (6) provides a valuable consistency indicator for the complex mode shapes, and indirectly for the inherent measured phase quantities. This is a key consideration in the damping correction problem because solution of the damped inverse problem is generally implied in the correction algorithm. If the nonnormal phase components of the normalized mode shapes X cannot satisfy Eq. (6), then the damping correction can introduce nonphysical quantities into the system stiffness and mass, adversely affecting the normal mode parameters we seek.

III. System Realization and the CBSI Procedure

In this section, a systematic procedure is reviewed for transforming a general first-order I/O model, such as that obtained from multiple-reference identification algorithms such as ERA or Polyreference, to an equivalent canonical variable form that yields the desired modal parameters. This procedure is termed the CBSI[4] algorithm. In the present context, CBSI effectively provides a first-order approximation of the true normal modal parameters, together with residual terms that are the result of nonproportional damping.

System realization algorithms such as ERA are used to compute a discrete-time model of the system that best approximates the measured Markov parameters, which are the discrete-time impulse responses of the system outputs with respect to the system inputs. Converting to continuous time, we obtain a nominal state-space realization, viz.,

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad y_d(t) = Cx(t) + Gu(t) \tag{7}$$

where, without loss of generality, we have assumed the output y_d to be displacement quantities. Furthermore, for displacement outputs, $G \approx 0$, and so the feedthrough term (which is not used in the present procedure) can be dropped. Contrasting Eq. (7) with Eq. (1), we can see that the $2n$ state vector $x(t)$ is a linear combination of the displacements q and velocities $\dot{q}$ of the underlying second-order structural model.

The system realization equation (7) can be decoupled into n pairs of complex conjugate modes by solving the eigenproblem of A. The resulting system exhibits the complex damped mode shapes of the inputs and outputs and the complex roots of the damped oscillations, viz.,

$$\begin{Bmatrix} \dot{z}_i(t) \\ \dot{\bar{z}}_i(t) \end{Bmatrix} = \begin{bmatrix} \sigma_i + j\omega_i & 0 \\ 0 & \sigma_i - j\omega_i \end{bmatrix} \begin{Bmatrix} z_i(t) \\ \bar{z}_i(t) \end{Bmatrix} + \begin{bmatrix} b_{zi} \\ \bar{b}_{zi} \end{bmatrix} u(t)$$

$$y(t) = \sum_i^n [c_{zi} \quad \bar{c}_{zi}] \begin{Bmatrix} z_i(t) \\ \bar{z}_i(t) \end{Bmatrix} \tag{8}$$

The nonnormalized complex damped mode shapes are given by the magnitude and phase of c_{zi}; hence, if all relative phase angles are 0 or π radians, the damping is classical (proportional):

$$\omega_{ni}^2 = \sigma_i^2 + \omega_i^2 \qquad \zeta_i = -(\sigma_i/\omega_{ni}) \tag{9}$$

An equivalent model in a modal displacement–velocity (MDV) basis is given as

$$\begin{Bmatrix} \dot{\eta}_i(t) \\ \ddot{\eta}_i(t) \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_{ni}^2 & -2\zeta_i\omega_{ni} \end{bmatrix} \begin{Bmatrix} \eta_i(t) \\ \dot{\eta}_i(t) \end{Bmatrix} + \begin{bmatrix} 0 \\ \phi_i^T \hat{B} \end{bmatrix} u(t)$$

$$y_d(t) = \sum_i^n [H\phi_i \quad 0] \begin{Bmatrix} \eta_i(t) \\ \dot{\eta}_i(t) \end{Bmatrix} \tag{10}$$

Here, $\hat{B}$ and H are input and output location influence arrays, ϕ_i and ω_{ni}^2 are the mode shapes and squared natural frequencies defined by the generalized undamped eigenproblem

$$K\Phi = M\Phi\Omega^2$$
$$\Phi^T M\Phi = I_{n\times n}$$
$$\Phi^T K\Phi = \Omega^2 = \mathrm{diag}\{\omega_{ni}^2\}, \qquad i = 1,\ldots,n \tag{11}$$
$$\Phi^T D\Phi = \Xi$$

and Ξ is the modal damping matrix. The CBSI algorithm[4] is used to transform Eq. (7) to its equivalent MDV basis form as follows. From the decoupled first-order damped modes equation (8), each mode i is transformed to its corresponding MDV form Eq. (10). To this end, the CBSI algorithm introduces the following transformation:

$$\begin{Bmatrix} \eta_i(t) \\ \dot{\eta}_i(t) \end{Bmatrix} = V_i \begin{Bmatrix} z_i(t) \\ \bar{z}_i(t) \end{Bmatrix} \tag{12}$$

This transformation can be determined as a product of two rotations and a scaling parameter:

$$V_i = \left(\frac{jd_i}{2\omega_i}\right)\begin{bmatrix} \sigma_i - j\omega_i & -1 \\ -\sigma_i - j\omega_i & 1 \end{bmatrix}\begin{bmatrix} -\sigma_i - r_i\omega_i & 1 \\ -\sigma_i^2 - \omega_i^2 & \sigma_i - r_i\omega_i \end{bmatrix} \tag{13}$$

where

$$r_i = \begin{bmatrix} \Im(b_{zi}) \\ \Re(b_{zi}) \end{bmatrix} \tag{14}$$

and d_i is a scalar quantity that will allow the eigenvector quantities to be mass-normalized, as in the MDV model Eq. (10). Applying Eqs. (13) and (14) to each complex mode of the realization, the CBSI-identified model is given as

$$\begin{Bmatrix} \dot{\xi}_l \\ \dot{\xi}_v \end{Bmatrix} = \begin{bmatrix} 0 & I \\ -\tilde{\Omega}^2 & -\tilde{\Xi} \end{bmatrix}\begin{Bmatrix} \xi_l \\ \xi_v \end{Bmatrix} + \begin{Bmatrix} B_1^T \\ \tilde{\Phi}_u^T \end{Bmatrix} u$$
$$y_d = [\tilde{\Phi}_y \quad H_2]\begin{Bmatrix} \xi_l \\ \xi_v \end{Bmatrix} \tag{15}$$

where ξ_l and ξ_v are the CBSI approximations to the modal displacements η and velocities $\dot{\eta}$ and $\tilde{\Phi}$, $\tilde{\Omega}^2$, and $\tilde{\Xi}$ are the approximations of the normal modal parameters Φ, Ω^2, and Ξ, respectively.

Note that, when $B_1 = 0$ and $H_2 = 0$, the CBSI model correctly captures the second-order model form using a diagonal damping matrix and requires no further corrections. In this case, the damping of the system is effectively proportional (at least from the standpoint of the available measurements) and the estimated modal parameters Φ, Ω^2, and Ξ can be taken as accurate. In the case where B_1 and H_2 are nonzero, however, it is clear that the state-space model is not the correct canonical form of the second-order modal equations. These nonzero residual quantities of the model are assumed to be due to the presence of nonproportional damping in the system response, which renders the assumption of a diagonal modal damping matrix (used in the CBSI transformation) invalid. In Sec. IV, a corrective transformation based on the residual quantity H_2 is developed to aid in the extraction of the correct normal modal parameters when nonproportional damping is present.

IV. Corrective Transformation for General Damping

In this section, a new transformation is developed to correct the deficiencies in the pseudonormal modes model provided by the CBSI transformation. This is necessary because the CBSI procedure assumes a one-to-one relationship between the damped and undamped modes, which is invalid in the case of nonproportional damping. Because CBSI is a similarity transformation that maintains the system equivalence, however, the CBSI model retains the out-of-phase components of the damped modes as residual quantities in the input and output influence matrices. These perturbations of the idealized model form therefore can be used to correct the deficiencies in the pseudonormal modes model.

We begin the present damping correction procedure with the model resulting from the CBSI procedure, Eq. (15). Any corrected displacement variable basis must be written as a linear combination of the state variables ξ_l and ξ_v, which are the approximate modal displacement and velocity variables of the CBSI model. We then define a new displacement basis for the model which is a perturbation of the estimated modal displacements, viz.,

$$\hat{\eta} = \xi_l + V_d\xi_v = [I \quad V_d]\begin{Bmatrix} \xi_l \\ \xi_v \end{Bmatrix} \tag{16}$$

Here $\hat{\eta}$ is a generalized displacement basis, i.e., it is a linear combination of the correct normal modal displacements η, but it does not necessarily diagonalize the stiffness and mass matrices of the new model. The $n\times n$ perturbation matrix V_d is the key to the transformation, for it defines the contribution of the estimated velocity variables ξ_v to the new displacement basis $\hat{\eta}$. The coupling of ξ_v with ξ_l, which are nearly opposite phase components of the measured response, is necessary to account for the complex phase quantities in the damped mode shapes of the model. From a practical standpoint, V_d will be computed to minimize or eliminate the residual quantities B_1 and H_2 in the transformed model.

To obtain a new model in a canonical basis, we must complement the displacement variables $\hat{\eta}$ with velocity variables $\dot{\hat{\eta}}$, which are definitively the time derivatives of the displacements. This can be accomplished by differentiating Eq. (16):

$$\dot{\hat{\eta}} = [I \quad V_d]\begin{Bmatrix} \dot{\xi}_l \\ \dot{\xi}_v \end{Bmatrix} \tag{17}$$

Then, substituting Eq. (17) into the equilibrium equation (15) and requiring that $B_1^T + V_d\tilde{\Phi}_u^T = 0$, the full $2n\times 2n$ damping correction transformation is obtained:

$$\begin{Bmatrix} \hat{\eta} \\ \dot{\hat{\eta}} \end{Bmatrix} = \begin{bmatrix} I & V_d \\ -V_d\tilde{\Omega}^2 & I - V_d\tilde{\Xi} \end{bmatrix}\begin{Bmatrix} \xi_l \\ \xi_v \end{Bmatrix} = \Psi\begin{Bmatrix} \xi_l \\ \xi_v \end{Bmatrix} \tag{18}$$

The state transformation equation (18) is now a function of the perturbation matrix V_d, which has yet to be determined. To properly determine V_d, we must consider the properties of the transformed model. Therefore, applying the change-of-basis given by Eq. (18) to both the equilibrium and output equations (15), we have

$$\begin{Bmatrix} \dot{\hat{\eta}} \\ \ddot{\hat{\eta}} \end{Bmatrix} = \Psi\begin{bmatrix} 0 & I \\ -\tilde{\Omega}^2 & \tilde{\Xi} \end{bmatrix}\Psi^{-1}\begin{Bmatrix} \hat{\eta} \\ \dot{\hat{\eta}} \end{Bmatrix} + \Psi\begin{bmatrix} B_1^T \\ \tilde{\Phi}_u^T \end{bmatrix} u$$
$$y_d = [\tilde{\Phi}_y \quad H_2]\begin{Bmatrix} \hat{\eta} \\ \dot{\hat{\eta}} \end{Bmatrix} \tag{19}$$

which leads to

$$\begin{Bmatrix} \dot{\hat{\eta}} \\ \ddot{\hat{\eta}} \end{Bmatrix} = \begin{bmatrix} 0 & I \\ -\hat{M}^{-1}\hat{K} & -\hat{M}^{-1}\hat{D} \end{bmatrix}\begin{Bmatrix} \hat{\eta} \\ \dot{\hat{\eta}} \end{Bmatrix} + \begin{bmatrix} \Theta_u^T \\ S\tilde{\Phi}_u^T - V_d\tilde{\Omega}^2\Theta_u^T \end{bmatrix} u$$
$$y_d = \left[\tilde{\Phi}_y - \Theta_y S^{-1}V_d\tilde{\Omega}^2 \quad \Theta_y S^{-1}\right]\begin{Bmatrix} \hat{\eta} \\ \dot{\hat{\eta}} \end{Bmatrix} \tag{20}$$

where

$$\Theta_u = B_1 + \tilde{\Phi}_u V_d^T \qquad \Theta_y = H_2 - \tilde{\Phi}_y V_d$$
$$S = (I - V_d\tilde{\Xi} + V_d\tilde{\Omega}^2 V_d) \tag{21}$$

and $\hat{M}$, $\hat{D}$, and $\hat{K}$ are to-be-determined mass, damping, and stiffness matrices with respect to the generalized displacement basis $\hat{\eta}$. For Eq. (20) to have a model form consistent with Eq. (10), the following constraints must be satisfied:

$$\Theta_u = B_1 + \tilde{\Phi}_u V_d^T = 0 \qquad \Theta_y = H_2 - \tilde{\Phi}_y V_d = 0 \tag{22}$$

The constraint equations (22) therefore provide the necessary equations for determining V_d. After determining V_d and applying the state transformation to obtain the model equation (20), the normal modal parameters can be obtained by solving the eigenproblem

$$(\hat{M}^{-1}\hat{K})T = T\Omega^2 \tag{23}$$

where $\hat{M}^{-1}\hat{K}$ is the lower left $n \times n$ partition of the state transition matrix in the new basis. Finally, assuming the constraint equations (22) are satisfied, the corrected model is transformed to the normal modes basis using $\hat{\eta} = T\eta$, viz.,

$$\begin{Bmatrix}\dot{\eta}\\ \ddot{\eta}\end{Bmatrix} = \begin{bmatrix}0 & I\\ -\Omega^2 & -\Xi\end{bmatrix}\begin{Bmatrix}\eta\\ \dot{\eta}\end{Bmatrix}\begin{bmatrix}0\\ \Phi_u^T\end{bmatrix}u \qquad y_d = [\Phi_y \quad 0]\begin{Bmatrix}\eta\\ \dot{\eta}\end{Bmatrix} \tag{24}$$

where

$$\Omega^2 = T^{-1}(\hat{M}^{-1}\hat{K})T \qquad \Xi = T^{-1}(\hat{M}^{-1}\hat{D})T$$
$$\Phi_u^T = T^{-1}(S\tilde{\Phi}_u^T) \qquad \Phi_y = (\tilde{\Phi}_y)T \tag{25}$$

The procedure is relatively straightforward, given that the matrix V_d can be determined. Unfortunately, this is not trivial even given that the constraint equations (22) are linear. In performing the damping correction procedure to obtain the proper second-order model form, it is critical to assess the physical properties of the model resulting from the present transformation. In particular, the relative phase information in the complex damped mode shapes may be due to effects other than nonproportional damping, such as nonlinearities, measurement noise, and numerical effects of the identification algorithm. In these cases, the transformation used to correct the model can become corrupted, and the new transformed model may not be consistent with the characteristic physics of the problem, leading to indefinite or complex matrices. We therefore have developed the following criterion for the transformation, which ensures, at least in part, the consistency of the transformed model with the underlying physics of the linear structural dynamics problem.

A. Displacement Consistency Criterion for V_d

We have developed the consistency criterion for V_d by examining the structural matrices of the corrected model as follows. Casting Eq. (15) into an equivalent symmetric form as in Eq. (2), converting to the corrected displacement velocity basis using Eq. (18), and multiplying through by Ψ^{-T}, we have

$$\begin{bmatrix}\hat{D} & \hat{M}\\ \hat{M} & 0\end{bmatrix}\begin{Bmatrix}\dot{\eta}\\ \ddot{\hat{\eta}}\end{Bmatrix} = \begin{bmatrix}-\hat{K} & 0\\ 0 & \hat{M}\end{bmatrix}\begin{Bmatrix}\hat{\eta}\\ \dot{\hat{\eta}}\end{Bmatrix} + \Psi^{-T}\begin{bmatrix}\tilde{\Xi}B_1^T + \tilde{\Phi}_u^T\\ B_1^T\end{bmatrix}u$$
$$y_d = [\tilde{\Phi}_y \quad H_2]\Psi^{-1}\begin{Bmatrix}\hat{\eta}\\ \dot{\hat{\eta}}\end{Bmatrix} \tag{26}$$

where

$$\begin{bmatrix}\hat{D} & \hat{M}\\ \hat{M} & 0\end{bmatrix} = \begin{bmatrix}0 & \hat{M}^{-1}\\ \hat{M}^{-1} & -\hat{M}^{-1}\hat{D}\hat{M}^{-1}\end{bmatrix}^{-1} = \left(\Psi\begin{bmatrix}0 & I\\ I & -\tilde{\Xi}\end{bmatrix}\Psi^T\right)^{-1}$$
$$\begin{bmatrix}-\hat{K} & 0\\ 0 & \hat{M}\end{bmatrix} = \begin{bmatrix}-\hat{K}^{-1} & 0\\ 0 & \hat{M}^{-1}\end{bmatrix}^{-1} = \left(\Psi\begin{bmatrix}-\tilde{\Omega}^{-2} & 0\\ 0 & I\end{bmatrix}\Psi^T\right)^{-1} \tag{27}$$

Then solving for the generalized model matrices, we obtain

$$\hat{K} = (\tilde{\Omega}^{-2} - V_dV_d^T)^{-1}$$
$$\hat{M} = \left[I - \tilde{\Xi}V_d^T - V_d\tilde{\Xi} + V_d(\tilde{\Xi}^2 - \tilde{\Omega}^2)V_d^T\right]^{-1}$$
$$\hat{D} = \hat{M}\left[\tilde{\Xi} + V_d(\tilde{\Xi}^3 - \tilde{\Omega}^2\tilde{\Xi} - \tilde{\Xi}\tilde{\Omega}^2)V_d^T - V_d(\tilde{\Xi}^2 - \tilde{\Omega}^2) - (\tilde{\Xi}^2 - \tilde{\Omega}^2)V_d^T\right]\hat{M} \tag{28}$$

subject to

$$V_d + V_d^T - V_d\tilde{\Xi}V_d^T = 0 \tag{29}$$

The condition given in Eq. (29) is analogous to Eq. (6) and is termed a displacement consistency criterion for the transformation V_d. This condition must be satisfied regardless of the other properties of the model. It is also possible to use Eq. (28) to constrain V_d by imposing conditions on the modified physical matrices, such as positive definiteness of the mass matrix and positive semidefiniteness of the stiffness and damping (symmetry is implicitly preserved). Combining the displacement consistency criteria with the transformation constraint equations (22), we can determine V_d such that the displacement sensor outputs are expressed solely as a function of the corrected modal displacements η and any unmeasured mode shape phase quantities are consistent with displacement variable assumptions.

B. Determination of V_d via Minimization

The constraint equations (22) provide a great deal of insight into the nature of the damping correction problem. For the overdetermined case, in which the number of independent sensors is greater than n, we may not be able to completely account for the out-of-phase component measure H_2 using the transformation V_d over the n identified modes. We can, however, directly determine a least-squares solution that will minimize $\|H_2 - \tilde{\Phi}_yV_d\|$. Thus, solutions that optimize (according to some measure) the incorporation of the mode shape phase relationships to determine corrected normal modes can be obtained directly using CBSI and the present procedure, rather than iteratively as in the methods of Zhang and Lallement[7,8] and Placidi et al.[9]

The underdetermined constraint problem, where the number of sensors is less than the number of modes, is much more difficult in that it admits an infinity of possible physical solutions that equally satisfy Eq. (22). An obvious choice is found using the uniquely defined Moore–Penrose pseudoinverse of $\tilde{\Phi}_y$, which yields a solution for V_d whose norm is minimized. This solution does not generally satisfy the displacement consistency criterion but can be used in an iterative fashion to seek a general solution that minimizes the magnitude of V_d while satisfying both of the constraint equations (22) and (29).

We can approach this quadratic matrix problem by noting that all solutions to Eq. (22) fit the form

$$V_d = V_p + Q_{n1}S_n^T \tag{30}$$

where

$$V_p = \tilde{\Phi}_y^+H_2 \qquad Q_{n1} = \mathrm{null}(\tilde{\Phi}_y)$$
$$S_n = S_n^{(0)} + (I - V_p\tilde{\Xi})^{-1}Q_{n1}S_{nn} \qquad S_n^{(0)} = -(I - V_p\tilde{\Xi})^{-1}V_pQ_{n1} \tag{31}$$

and S_{nn} is subject to

$$S_{nn}^TN_1 + N_1^TS_{nn} - S_{nn}^TN_2S_{nn} + R = 0 \tag{32}$$

where

$$N_1 = Q_{n1}^T(I - V_p\tilde{\Xi})^{-T}(Q_{n1} - \tilde{\Xi}S_n^{(0)})$$
$$N_2 = Q_{n1}^T(I - V_p\tilde{\Xi})^{-T}\tilde{\Xi}(I - V_p\tilde{\Xi})^{-1}Q_{n1} \tag{33}$$
$$R = S_n^{(0)T}Q_{n1} + Q_{n1}^TS_n^{(0)} - S_n^{(0)T}\tilde{\Xi}S_n^{(0)}$$

Thus we must solve for a scaling matrix S_{nn} of dimension $(n - l)$, where n is the number of measured modes and l is the number of measured sensors. Although this problem is reduced in scale as compared to Eq. (29), it is still difficult to solve in a direct fashion.

A two-stage iterative procedure has been developed to seek a solution to Eq. (32). Utilizing the sequential quadratic programming (SQP)[13] method to handle the nonlinearity introduced by the constraint, we project Eq. (32) onto the singular values of the residual matrix R and determine the projection of S_{nn}, a vector, which satisfies the scalar constraint and minimizes the norm of the solution vector. The solution for S_{nn} then is incorporated into S_n and new matrices N_1 and R are found using Eq. (33). The two-stage procedure concludes when the norm of the residual R is effectively zero. Details of the procedure can be found in the Appendix.

V. Implementation and Numerical Examples

The present damping correction algorithm, along with the general CBSI procedure outlined in Sec. II, has been implemented for the following numerical examples to demonstrate the procedure.

The first example is a three-degree-of-freedom (DOF) mass-spring system with nonproportional damping. The system equations of motion are given as

$$\begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{q}_3 \end{Bmatrix} + \begin{bmatrix} 2 & -0.1 & -0.1 \\ -0.1 & 0.2 & -0.1 \\ -0.1 & -0.1 & 1.1 \end{bmatrix} \begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{Bmatrix} + \begin{bmatrix} 4 & -2 & 0 \\ -2 & 2 & -2 \\ 0 & -2 & 22 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ u \\ 0 \end{Bmatrix} \tag{34}$$

The resultant undamped modal parameters then are given as the eigenvalues and eigenvectors of the mass and stiffness matrices, viz.,

$$\Omega^2 = \begin{bmatrix} 1.8980 & 0 & 0 \\ 0 & 5.8798 & 0 \\ 0 & 0 & 22.222 \end{bmatrix}, \quad \Phi = \begin{bmatrix} 0.6875 & 0.7261 & -0.0121 \\ 0.7226 & -0.6824 & 0.1104 \\ 0.0719 & -0.0847 & -0.9938 \end{bmatrix} \tag{35}$$

and the nonproportionality of the damping can be verified by computation of the modal damping matrix, given as

$$\Phi^T D \Phi = \begin{bmatrix} 0.9359 & 0.8991 & 0.0534 \\ 0.8991 & 1.2552 & 0.0562 \\ 0.0534 & 0.0562 & 1.1090 \end{bmatrix} \tag{36}$$

A balanced state-space realization of Eq. (34) for displacement sensing, similar to that obtained from ERA, can be determined from test-measured impulse response functions. Applying the symmetric form of CBSI to the resultant realization yields

$$\tilde{\Omega}^2 = \begin{bmatrix} 2.5043 & 0 & 0 \\ 0 & 4.4578 & 0 \\ 0 & 0 & 22.2155 \end{bmatrix}, \quad \tilde{\Xi} = \begin{bmatrix} 1.2281 & 0 & 0 \\ 0 & 0.9636 & 0 \\ 0 & 0 & 1.1083 \end{bmatrix}$$

$$\tilde{\Phi}_y = \begin{bmatrix} 0.9252 & 0.7416 & -0.0165 \\ 0.9025 & -0.8288 & 0.1109 \\ 0.0907 & -0.1071 & -0.9937 \end{bmatrix}, \quad H_2 = \begin{bmatrix} -0.1769 & 0.3085 & 0.0039 \\ 0.2784 & 0.2180 & -0.0004 \\ 0.0376 & 0.0225 & -0.0001 \end{bmatrix} \tag{37}$$

Using Eq. (29), we can verify that the displacement consistency criterion is satisfied. Then, assuming three sensors measure q_1, q_2, and q_3, from Eq. (22) the unique solution V_d is found to be

$$V_d = \begin{bmatrix} 0.0418 & 0.2907 & 0.0020 \\ -0.2907 & 0.0533 & 0.0027 \\ -0.0027 & -0.0018 & 0.0000 \end{bmatrix} \tag{38}$$

Completing the correction procedure by forming Ψ from Eq. (18), the new modal parameters are equivalent to Eqs. (35) and (36).

For the underdetermined case, if we restrict the problem by using only the output at q_2 and applying the iterative minimization approach, we obtain

$$V_d = \begin{bmatrix} 0.0413 & 0.2882 & -0.0188 \\ -0.2878 & 0.0524 & -0.0199 \\ 0.0233 & 0.0123 & 0.0004 \end{bmatrix} \tag{39}$$

and the resultant modal parameters are

$$\Omega^2 = \begin{bmatrix} 1.8911 & 0 & 0 \\ 0 & 5.7811 & 0 \\ 0 & 0 & 22.6849 \end{bmatrix}, \quad \Phi_m = \begin{bmatrix} 0.7185 & -0.6854 & 0.1185 \end{bmatrix}, \quad \Xi = \begin{bmatrix} 0.9394 & 0.8643 & -0.5025 \\ 0.8643 & 1.2076 & -0.4212 \\ -0.5025 & -0.4212 & 1.1530 \end{bmatrix} \tag{40}$$

where Φ_m is the measured partition of the corrected mode shapes. By inspection, the damping correction using only one sensor was partially successful. The accuracy of the eigenvalues and eigenvectors for the lowest two modes was improved over the CBSI estimate Eq. (37), but the highest mode was somewhat adversely affected. This is because, as the number of measurements becomes relatively small as compared to the number of measured modes, we have progressively fewer mode shape phase quantities to guide the damping correction, and the minimization approach may not yield the best solution.

An improved result can be obtained, however, by restricting the number of modes that are coupled through the transformation. For example, by examining the relative magnitudes of $\tilde{\Phi}_y$ and H_2 for mode 3, or alternatively looking at the modal phase collinearity (MPC)[14] for mode 3, it is evident that mode 3 is nearly proportionally damped and is effectively decoupled from modes 1 and 2. Therefore, operating only on modes 1 and 2, V_d is given as

$$V_d = \begin{bmatrix} 0.0417 & 0.2904 & 0 \\ -0.2904 & 0.0532 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{41}$$

and the resultant corrected modal parameters are

$$\Omega^2 = \begin{bmatrix} 1.8998 & 0 & 0 \\ 0 & 5.8761 & 0 \\ 0 & 0 & 22.2155 \end{bmatrix}, \quad \Phi_y = [0.7230 \quad -0.6818 \quad 0.1109], \quad \Xi = \begin{bmatrix} 0.9365 & 0.8978 & 0 \\ 0.8978 & 1.2552 & 0 \\ 0 & 0 & 1.1083 \end{bmatrix} \tag{42}$$

which leads to a better approximation of modes 1 and 2 without altering the reasonable estimate of mode 3 obtained by CBSI.

A second numerical example shown in Fig. 1 is of a 36-DOF planar truss with nonproportional damping. Figure 2 shows the exact nonproportional damping matrix assumed for the model. In Fig. 3, an example of a mode shape highly affected by the nonproportional damping is shown. Using 18 available sensors, the CBSI-estimated modal parameters were successfully corrected. Table 1 compares various implementations of the present procedure using modal set selection with overdetermined, well-determined, and underdetermined solutions for V_d. The comparisons are made in terms of the modal assurance criteria (MAC)[15] between the derived mode shape at the sensor locations and the known exact mode shape. The MPC also is shown as an indicative measure of the complexity of the damped mode shapes. The standard technique for estimating the real mode shape from the damped complex mode shape is to use the magnitude of each component of the complex vector and determine

Table 1 Normal mode shape estimates for 36-DOF nonproportionally damped truss

		MAC: identified mode shape vs exact				
Mode No.	MPC	Standard technique	CBSI SYM	12 Modes (overdetermined)	28 Modes (undetermined)	36 Modes (undetermined)
16	0.9887	0.9993	0.9995	0.9998	1.0000	0.9994
17	0.9866	0.9993	0.9994	0.9998	1.0000	0.9993
19	0.9481	0.9983	0.9986	0.9993	0.9986	0.9994
20	0.9743	0.9991	0.9994	1.0000	1.0000	0.9997
22	0.9771	0.9617	0.9444	0.9996	0.9964	0.9947
23	0.9748	0.9617	0.9410	0.9997	0.9961	0.9945
25	0.9755	0.9998	0.9999	1.0000	0.9995	0.9992
26	0.8468	0.9934	0.9980	1.0000	0.9832	0.9814
27	0.5722	0.8755	0.9241	0.9999	0.9822	0.9786
28	0.5990	0.8896	0.9299	0.9999	0.9955	0.9696
29	0.9404	0.9964	0.9986	1.0000	1.0000	0.9697
30	0.9853	0.9996	1.0000	1.0000	1.0000	1.0000

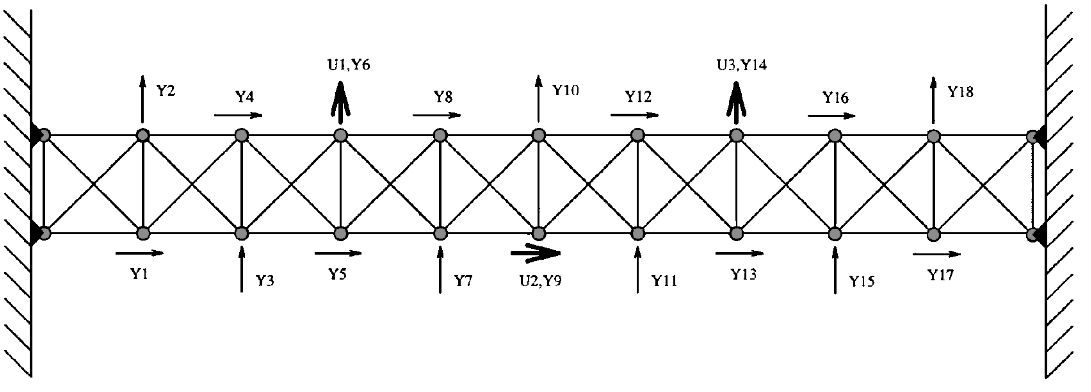

Fig. 1 Numerical planar truss example.

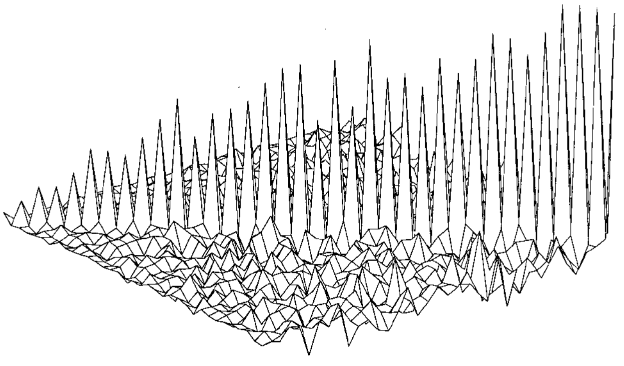

Fig. 2 Exact nonproportional modal damping for truss example.

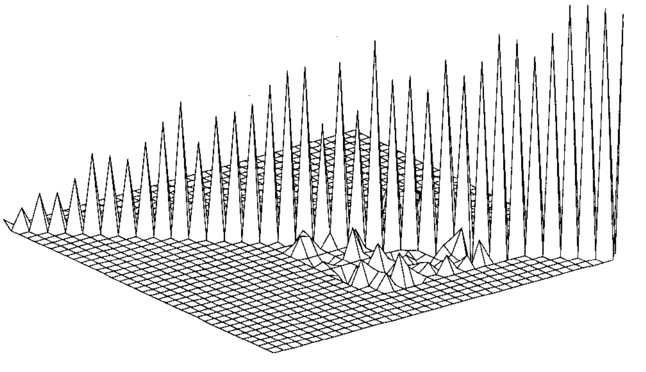

Fig. 4 Corrected modal damping matrix for truss example.

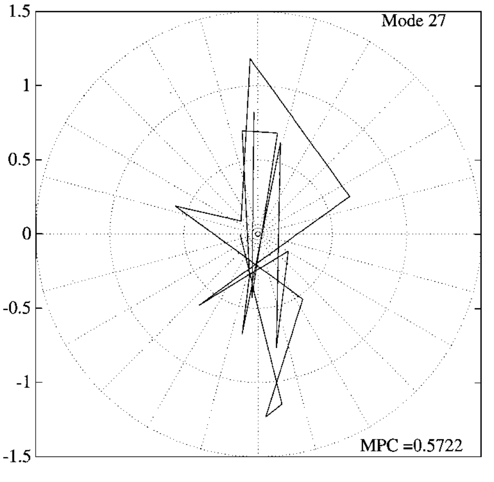

Fig. 3 Polar plot of complex mode shape with low modal phase collinearity: damped mode shape mag vs phase.

the corresponding sign by rounding off the phase angle to $\pm\pi/2$ radians. The accuracy of these mode shape estimates is shown in Table 1, together with the results for the CBSI-estimated mode shapes. Note that, even without using the damping corrective transformation, the CBSI transformation theory-based mode shape estimates are generally superior to the standard technique, especially for the most complex of the modes.

Using the CBSI model together with the present damping corrective transformation, the best solution for the normal modes is obtained through selection of the 12 modes possessing the lowest MPC measures, resulting in an overdetermined solution for the matrix V_d. Figure 4 shows the resultant modal damping matrix with off-diagonal terms determined from the correction procedure. In Fig. 5, the normal mode shape estimates for mode 27 are compared. The last two columns in Table 1 show that reasonable and improved results also are obtained when 28 modes and, finally, all 36 modes are simultaneously adjusted using only the 18 sensors. These cases lead to an underdetermined solution for V_d, which requires that the displacement consistency criteria be explicitly met, while minimizing the norm of V_d through the SQP-based iterative minimization.

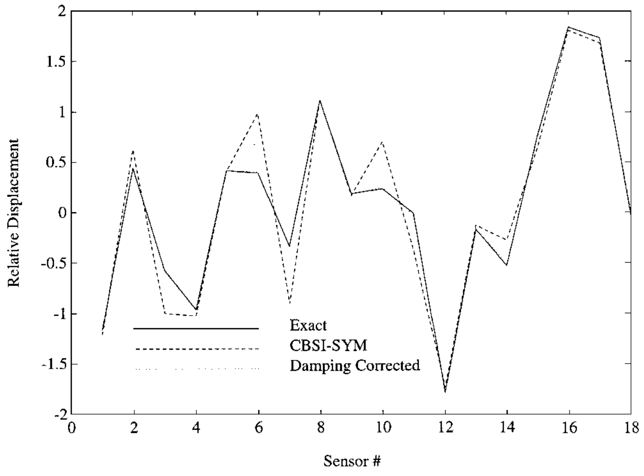

Fig. 5 Improvement in truss mode shape due to damping corrective transformation: mode 27.

VI. Concluding Remarks

A general damping correction procedure for determining the undamped modal parameters from damped system realization parameters is presented. The procedure begins with the assumption that the system may be proportionally damped and obtains estimated normal modes via the CBSI algorithm. If the displacement and velocity modes are different, the damping correction procedure is invoked to extract the normal modes and associated nonproportional damping matrix. This corrective transformation can be carried out in two distinct ways. If the number of sensors is at least equal to the number of identified modes, the transformation can be obtained by solving a linear constraint equation. If the number of sensors is less than the number of identified modes, the linear constraint is underdetermined and the transformation must be determined either directly or iteratively such that a quadratic matrix consistency criterion is satisfied. The present damping correction procedure is demonstrated through numerical examples.

The application of any damping extraction algorithm to experimentally identified damped modes is problematic because of the effects of noise, numerical algorithms for curve fitting, and truncation of high-frequency modes that are coupled to the modes being identified. The present procedure offers a number of advantages over existing techniques for handling such experimental effects. First, the displacement consistency criterion provides a constraint on the coupling transformation that helps to ensure the physical consistency of the transformation in the presence of experimental error. Second, in the case of an overdetermined set of equations, the linearity of the governing equation avoids the convergence problems of an iterative solution. Finally, the use of the CBSI algorithm leads to an initial real-valued model that yields an estimate of the desired normal modes and segregates the effects of the modal phase quantities into residual quantities that the damping procedure then utilizes to correct the model. Because these residual quantities are subject to significant relative errors in the case of experimental data, the development of a robust and reliable algorithm for experimental data will require the application of weighting factors or other numerical strategies to reflect the relative confidence in the quantities used for damping correction. We hope to report on these developments in the near future.

Appendix: Two-Stage Iterative Procedure to Solve S_{nn}

A two-stage iterative procedure has been developed to seek a solution to S_{nn}. For the inner stage, we first project Eq. (32) onto the singular values of the residual matrix R, viz.,

$$R = \sum^{n-l} s_i p_i p_i^T \tag{A1}$$

$$x_i^T b_i + b_i^T x_i - x_i^T N_2 x_i + s_i = 0$$
$$x_i = S_{nn} p_i \qquad b_i = N_1 p_i \tag{A2}$$

Then, for $i = 1$ to $(n - l)$, we minimize $\|x_i\|$ subject to Eq. (A2). Given the nonlinearity of the constraint, we can apply the SQP method[13]

$$\min_{x_i, \lambda} \bar{J} = \tfrac{1}{2} x_i^T x_i - \lambda \Phi_i$$
$$\Phi_i(x_i) = x_i^T b_i + b_i^T x_i - x_i^T N_2 x_i + s_i \tag{A3}$$

leading to the system of equations

$$\begin{bmatrix} W^{(k)} & -\left(\dfrac{\partial \Phi_i^{(k)}}{\partial x_i}\right)^T \\ -\left(\dfrac{\partial \Phi_i^{(k)}}{\partial x_i}\right) & 0 \end{bmatrix} \begin{Bmatrix} \delta x_i \\ \lambda^{(k)} \end{Bmatrix} = \begin{bmatrix} -x_i^{(k)} \\ \Phi_i^{(k)} \end{bmatrix} \tag{A4}$$

where

$$W^{(k)} = I + 2\lambda^{(k)} N_2 \qquad \frac{\partial \Phi_i^{(k)}}{\partial x_i} = 2\left(b_i - N_2 x_i^{(k)}\right)^T$$
$$\Phi_i^{(k)} = \Phi_i\left(x_i^{(k)}\right) \qquad x_i^{(k+1)} = x_i^{(k)} + \delta x_i \tag{A5}$$
$$x_i^{(0)} = 0 \qquad \lambda^{(0)} = 0$$

Each solution x_i then is projected back to form S_{nn}, viz.

$$S_{nn} = \sum^{n-l} x_i p_i^T \tag{A6}$$

The procedure requires an outer iterative stage because, in determining the ith projection of the solution x_i, we do not constrain the projection vectors p_i to remain orthogonal through the matrix equation (32). Hence, after determining S_{nn} from Eq. (A6) at outer iteration j, we incorporate the solution to determine

$$S_n^{(j+1)} = S_n^{(j)} + (I - V_p \tilde{\Xi})^{-1} Q_{n1} S_{nn}^{(j)} \tag{A7}$$

and form new matrices N_1 and R using $S_n^{(j+1)}$. The outer iteration concludes when the norm of the residual R is within a specified bound ε.

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